*On the Solutions of Linear Homogeneous Difference Equations.**

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Introduction.

Let us consider the linear homogeneous difference equation of order n,

$$F(x+n)+\bar{a}_1(x)F(x+n-1)+\bar{a}_2(x)F(x+n-2)+\dots+\bar{a}_n(x)F(x)=0, \quad (1)$$

in which the coefficients are analytic at infinity or have poles there. By means of a transformation of the form

$$F(x)=x^{\mu x}f(x),$$

we arrive at the new equation,

$$f(x+n)+a_1(x)f(x+n-1)+a_2(x)f(x+n-2)+\dots+a_n(x)f(x)=0, \quad (2)$$

where

$$a_k(x)=(x+n-k)^{\mu(x+n-k)}(x+n)^{-\mu(x+n)}\bar{a}_k(x), \qquad k=1, 2, \dots, n.$$

It is obvious that integers μ exist such that each of the functions $a_k(x)$ is analytic at infinity. For the value of μ we choose the least integer such that the functions $a_k(x)$ are all analytic at infinity.† Let us write

$$a_k(x)=a_{k0}+\frac{a_{k1}}{x}+\frac{a_{k2}}{x^2}+\dots, \qquad k=1, 2, \dots, n, \ |x|\geqq R. \quad (3)$$

We shall confine our attention to the case in which the constants $a_{10}, a_{20}, \dots, a_{n0}$ are such that the *characteristic algebraic equation* ‡

$$\alpha^n+a_{10}\alpha^{n-1}+a_{20}\alpha^{n-2}+\dots+a_{n0}=0, \quad (4)$$

has its roots $\alpha_1, \alpha_2, \dots, \alpha_n$, different from each other and from zero. In this

* Read before the American Mathematical Society at Chicago, December 29, 1914.

† It is easy to see that $x^{-\mu k}\bar{a}_k(x)$ is analytic at infinity for every value of k from 1 to n and that μ is the least integer for which this is true. We shall say that the value of $x^{-\mu k}\bar{a}_k(x)$ at infinity is $\bar{a}_{k0}$. Comparing with equation (3) we see that $a_{k0}=e^{-\mu k}\bar{a}_{k0}$.

‡ It will be said that this is the characteristic algebraic equation *associated* either with (2) or with (1).

case, as is well-known, equation (2) has n formal power series solutions of the form *

$$f_i(x)=\alpha_i^x x^{\mu_i}\left(1+\frac{c_{i1}}{x}+\frac{c_{i2}}{x^2}+\ldots.\right),\ i=1,\ 2,\ \ldots.,\ n. \tag{5}$$

These are in general divergent.

In this paper I shall show how the formal solutions (5) afford a means by which equation (2) may be separated into two members so that the method of successive approximations, when applied in its usual form, will lead to an actual solution of (2) and by aid of this single solution shall develop means by which the most important fundamental systems of solutions of (2), and hence of (1), are found.

This paper contains no new results, so that its novelty lies entirely in the new methods employed. The theorems demonstrated are essentially equivalent to those obtained by Birkhoff in the *Transactions of the American Mathematical Society*, Vol. XII (1911), pp. 242-284, and hence are more complete than the similar ones previously demonstrated by me and published in the volume just referred to, pp. 99-134. The method of the present paper is more closely related to that of my previous paper than to that of Birkhoff's later memoir. It shares with both of them the desirable property of being a direct means for attaining the end in view; but it is essentially different from either of them and in several respects is to be preferred to them. I have found it convenient to use the Birkhoff modification of the integral earlier employed by me, since this modified integral serves the purpose of securing a rather simpler fundamental system of solutions, especially as regards the asymptotic character of the functions in the solution.

For other methods of dealing with the problem of difference equations see Nörlund, *Acta Mathematica*, Vol. XL, pp. 191–249, and the papers to which he refers.

§1. *Formal Solutions by Successive Approximation.*

Let us denote by $l_i(x)$ the functions

$$l_i(x)=\alpha_i^x x^{\mu_i}\left(1+\frac{c_{i1}}{x}+\frac{c_{i2}}{x^2}+\ldots.+\frac{c_{im}}{x^m}\right),\ i=1,\ 2,\ \ldots.,\ n.$$

Form the equation

* The value of the constants μ_i, c_{i1}, c_{i2}, ... may be determined by substituting the value of $f_i(x)$ from (5) into (2) and directly reckoning out the constants so that the resulting equation shall be a formal identity in x.

$$\begin{vmatrix} g_1(x+n) & g_1(x+n-1)\dots\dots g_1(x) \\ l_1(x+n) & l_1(x+n-1)\dots\dots l_1(x) \\ l_2(x+n) & l_2(x+n-1)\dots\dots l_2(x) \\ \dots\dots & \dots\dots\dots\dots \\ l_n(x+n) & l_n(x+n-1)\dots\dots l_n(x) \end{vmatrix} = 0. \tag{6}$$

This is obviously a linear homogeneous difference equation of order n at most; and it is of order n if the minors of $g_1(x+n)$ and $g_1(x)$ are neither identically zero. These minors are $D(x)$ and $D(x+1)$ respectively, where

$$D(x) = \begin{vmatrix} l_1(x+n-1) & l_1(x+n-2)\dots\dots l_1(x) \\ l_2(x+n-1) & l_2(x+n-2)\dots\dots l_2(x) \\ \dots\dots & \dots\dots\dots\dots \\ l_n(x+n-1) & l_n(x+n-2)\dots\dots l_n(x) \end{vmatrix}. \tag{7}$$

Divide the first row of this determinant by $\alpha_1^x x^{\mu_1}$, the second row by $\alpha_2^x x^{\mu_2}, \dots\dots$, the last by $\alpha_n^x x^{\mu_n}$. Each element of the resulting determinant is analytic at infinity; and hence the function $\bar{D}(x)$ represented by this resulting determinant is analytic at infinity. It is easy to see that the value of $\bar{D}(x)$ at infinity is

$$\bar{D}(\infty) = \begin{vmatrix} \alpha_1^{n-1} & \alpha_1^{n-2}\dots\dots \alpha_1 & 1 \\ \alpha_2^{n-1} & \alpha_2^{n-2}\dots\dots \alpha_2 & 1 \\ \dots\dots & \dots\dots\dots & . \\ \alpha_n^{n-1} & \alpha_n^{n-2}\dots\dots \alpha_n & 1 \end{vmatrix};$$

or

$$\bar{D}(\infty) = \prod_{r,s} (\alpha_r - \alpha_s), \quad r < s.$$

Since the numbers $\alpha_1, \alpha_2, \dots\dots, \alpha_n$ are all different by hypothesis, we see that $\bar{D}(x)$ is not identically zero; in fact, it is different from zero at infinity. Hence, $D(x)$ is not identically zero: and therefore equation (6) is of order n.

In the determinant in equation (6) divide the second row by $\alpha_1^x x^{\mu_1}$, the third row by $\alpha_2^x x^{\mu_2}, \dots\dots$, the last row by $\alpha_n^x x^{\mu_n}$, and expand the resulting determinant in terms of the elements of the first row. The coefficients of the quantities $g_1(x+n)$, $g_1(x+n-1)$, $\dots\dots$, $g_1(x)$ are obviously all analytic at infinity, that of $g_1(x+n)$, namely, $\bar{D}(x)$, being different from zero at infinity. Hence we have from (6) an equation of the form

$$g_1(x+n) + \lambda_1(x) g_1(x+n-1) + \lambda_2(x) g_1(x+n-2) + \dots\dots + \lambda_n(x) g_1(x) = 0, \tag{8}$$

in which $\lambda_1(x)$, $\lambda_2(x)$, $\dots\dots$, $\lambda_n(x)$ are analytic at infinity. This equation has the solutions $l_1(x)$, $l_2(x)$, $\dots\dots$, $l_n(x)$, as we see readily from equation (6). Moreover, these form a fundamental system of solutions, since $D(x)$ is not identically zero.

By means of the fact that equation (8) has the solution $l_1(x), l_2(x), \ldots, l_n(x)$ and the relation which exists between $l_i(x)$, $i=1, 2, \ldots, n$, and the formal solution (5) of equation (2), it may be shown by direct computation that

$$\lambda_i(x) = a_{i0} + \frac{a_{i1}}{x} + \ldots + \frac{a_{i,\,m+1}}{x^{m+1}} + \frac{a_{i,\,m+2}}{x^{m+2}} + \ldots;$$

that is, that the expansion of $\lambda_i(x)$ in a descending power series in x coincides with that of $a_i(x)$ as far as the term in $1/x^{m+1}$.

Now, write equation (2) in the form

$$\sum_{k=0}^{n} \lambda_k(x) f(x+n-k) = \sum_{k=1}^{n} \Psi_k(x) f(x+n-k), \qquad \lambda_0(x)=1. \tag{9}$$

Then the function $\Psi_k(x)$ may be expanded as follows:

$$\Psi_k(x) = \frac{\beta_{k,\,m+2}}{x^{m+2}} + \frac{\beta_{k,\,m+3}}{x^{m+3}} + \ldots, \qquad k=1, 2, \ldots, n. \tag{10}$$

Form the system of equations

$$\left.\begin{array}{l} \sum_{k=0}^{n} \lambda_k(x) g_1(x+n-k) = 0, \\ \sum_{k=0}^{n} \lambda_k(x) g_2(x+n-k) = \sum_{k=1}^{n} \Psi_k(x) g_1(x+n-k), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \sum_{k=0}^{n} \lambda_k(x) g_i(x+n-k) = \sum_{k=1}^{n} \Psi_k(x) g_{i-1}(x+n-k), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \tag{11}$$

By means of these, used as equations for successive approximation, we shall be able to obtain formal solutions of equation (9) or (2).

The first equation in (11) is homogeneous and has the fundamental system of solutions $l_1(x), l_2(x), \ldots, l_n(x)$, as we saw above. The remaining equations in (11) are of the form

$$\sum_{k=0}^{n} \lambda_k(x) g(x+n-k) = \eta(x). \tag{12}$$

We shall next find two formal solutions of this typical equation. Our method is the classic one of variation of parameters. We assume a solution of the form

$$g(x) = \sum_{k=1}^{n} t_k(x) l_k(x), \tag{13}$$

and determine the functions $t_1(x), t_2(x), \ldots, t_n(x)$ in a convenient way so that they shall formally satisfy equation (12). Thus we may write

$$g(x+1)=\sum_{k=1}^{n} t_k(x)l_k(x+1), \quad \text{if} \quad \sum_{k=1}^{n} \Delta t_k(x)l_k(x+1)=0;$$
$$g(x+2)=\sum_{k=1}^{n} t_k(x)l_k(x+2), \quad \text{if} \quad \sum_{k=1}^{n} \Delta t_k(x)l_k(x+2)=0;$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$$
$$g(x+n-1)=\sum_{k=1}^{n} t_k(x)l_k(x+n-1), \quad \text{if} \quad \sum_{k=1}^{n} \Delta t_k(x)l_k(x+n-1)=0;$$
$$g(x+n)=\sum_{k=1}^{n} t_k(x)l_k(x+n)+\sum_{k=1}^{n} \Delta t_k(x)l_k(x+n).$$

So far we have imposed $n-1$ conditions on the n functions $t_1(x), t_2(x), \ldots,$ $t_n(x)$. We obtain an n-th condition, namely,

$$\sum_{k=1}^{n} \Delta t_k(x)l_k(x+n)=\eta(x),$$

by substituting the preceding values of $g(x), g(x+1), \ldots, g(x+n)$ in equation (12), remembering that $l_1(x), l_2(x), \ldots, l_n(x)$ satisfy the equation obtained from (12) by replacing $\eta(x)$ by zero.

The n equations of condition now placed upon the functions $t(x)$ may readily be solved by determinants for the quantities $\Delta t_k(x)$, $k=1, 2, \ldots, n$. Thus we have

$$\Delta t_k(x)=(-1)^{n+k}\eta(x)M_k(x+1)/M(x+1), \qquad k=1, 2, \ldots, n, \tag{14}$$

where

$$M(x)=\begin{vmatrix} l_1(x) & l_2(x) & \ldots l_n(x) \\ l_1(x+1) & l_2(x+1) & \ldots l_n(x+1) \\ \cdots & \cdots & \cdots \\ l_1(x+n-1) & l_2(x+n-1) & \ldots l_n(x+n-1) \end{vmatrix}$$

and $M_k(x)$ is the minor of the element $l_k(x+n-1)$ in $M(x)$.

Now, the function

$$M(x)/l_1(x)l_2(x)\ldots l_n(x)$$

is analytic at infinity and is different from zero at that point, as one may see readily by considering its value in the form of a determinant obtained from that for $M(x)$ by dividing the columns in order by $l_1(x), l_2(x), \ldots, l_n(x)$; for each element of the resulting determinant is analytic at infinity and the determinant has at infinity the same value as the determinant $\overline{D}(x)$ considered above. Likewise it may be shown that the function

$$M_k(x)/l_1(x)\ldots l_{k-1}(x)l_{k+1}(x)\ldots l_n(x)$$

is analytic at infinity and is different from zero at that point. Hence, we may write

$$(-1)^{n+k}M_k(x)/M(x)=A_k(x)/l_k(x),$$

where $A_k(x)$ is analytic at infinity and is different from zero at that point. Hence, we have from (14)

$$\Delta t_k(x)=\eta(x)A_k(x+1)/l_k(x+1), \qquad k=1, 2, \ldots, n. \tag{15}$$

Equations (15) have the following two formal solutions:

$$t_k(x)=-\sum_{i=0}^{\infty}\eta(x+i)A_k(x+i+1)/l_k(x+i+1), \qquad k=1, 2, \ldots, n;$$

$$t_k(x)=\sum_{i=1}^{\infty}\eta(x-i)A_k(x-i+1)/l_k(x-i+1), \qquad k=1, 2, \ldots, n.$$

Substituting these values in (13), we have the following two particular formal solutions of equation (12):

$$g^+(x)=-\sum_{i=0}^{\infty}\sum_{k=1}^{n}\eta(x+i)A_k(x+i+1)l_k(x)/l_k(x+i+1); \tag{16}$$

$$g^-(x)=\sum_{i=1}^{\infty}\sum_{k=1}^{n}\eta(x-i)A_k(x-i+1)l_k(x)/l_k(x-i+1). \tag{17}$$

These formal solutions will be denoted by

$$S_x(\eta) \quad \text{and} \quad T_x(\eta),$$

respectively.

Let $p_1(x)$, $p_2(x)$, $\ldots$, $p_n(x)$ be n arbitrary periodic functions of x of period 1. Then for the formal solutions of the successive equations in (11) we may conveniently select either of the following sets:

$$g_1^+(x)=\sum_{k=1}^{n}p_k(x)l_k(x),$$

$$g_2^+(x)=\sum_{k=1}^{n}p_k(x)l_k(x)+S_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)\sum_{k=1}^{n}p_k(x)l_k(x+n-i_1)\},$$

$$g_3^+(x)=\sum_{k=1}^{n}p_k(x)l_k(x)+S_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)\sum_{k=1}^{n}p_k(x)l_k(x+n-i_1)\}$$
$$+S_x[\sum_{i_2=1}^{n}\Psi_{i_2}(x)S_{x+n-i_2}\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)\sum_{k=1}^{n}p_k(x)l_k(x+n-i_1)\}],$$

. . . . ;

$$g_1^-(x)=\sum_{k=1}^{n}p_k(x)l_k(x),$$

$$g_2^-(x)=\sum_{k=1}^{n}p_k(x)l_k(x)+T_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)\sum_{k=1}^{n}p_k(x)l_k(x+n-i_1)\},$$

$$g_3^-(x)=\sum_{k=1}^{n}p_k(x)l_k(x)+T_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)\sum_{k=1}^{n}p_k(x)l_k(x+n-i_1)\}$$
$$+T_x[\sum_{i_2=1}^{n}\Psi_{i_2}(x)T_{x+n-i_2}\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)\sum_{k=1}^{n}p_k(x)l_k(x+n-i_1)\}],$$

. . . .

We are thus led to the following two formal expansions:

$$f^{+}(x)=\sum_{k=1}^{n}p_k(x)l_k(x)+S_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)\sum_{k=1}^{n}p_k(x)l_k(x+n-i_1)\}$$
$$+S_x[\sum_{i_2=1}^{n}\Psi_{i_2}(x)S_{x+n-i_2}\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)\sum_{k=1}^{n}p_k(x)l_k(x+n-i_1)\}]+\ldots., \quad (18)$$

$$f^{-}(x)=\sum_{k=1}^{n}p_k(x)l_k(x)+T_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)\sum_{k=1}^{n}p_k(x)l_k(x+n-i_1)\}$$
$$+T_x[\sum_{i_2=1}^{n}\Psi_{i_2}(x)T_{x+n-i_2}\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)\sum_{k=1}^{n}p_k(x)l_k(x+n-i_1)\}]+\ldots.. \quad (19)$$

It is easy to verify that these functions afford formal solutions of equation (9). It is sufficient to this purpose to make a direct substitution of the expansions into the equation, reducing the result by means of the relations

$$\sum_{k=0}^{n}\lambda_k(x)S_{x+n-k}(\eta)=\eta,\quad \sum_{k=0}^{n}\lambda_k(x)T_{x+n-k}(\eta)=\eta.$$

Each of these solutions is in general illusory on account of the divergence of certain series contained in the terms of the expansions. Let α_r and α_s be any two α's such that $|\alpha_r|>|\alpha_s|$. Put $p_i(x)=0$, $i\neq r$; $p_r(x)=1$. Then the second term in the series in (18) takes the form

$$S_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_r(x+n-i_1)\}=-\sum_{i=0}^{\infty}\sum_{k=1}^{n}\Psi_{i_1}(x+i)A_k(x+i+1)$$
$$l_r(x+n+i-i_1)l_k(x)/l_k(x+i+1).$$

It is easy to see that the series in the second member of this equation is not in general convergent. A similar discussion may be made of the second term in the series in (21); it turns out that the series denoted by this term is not in general convergent.

§ 2. *Two Actual Solutions.*

From the last result of the preceding section we see that the series in (18) and (19) do not in general represent actual solutions of equation (9). We shall now show that an appropriate choice of the periodic functions $p_1(x)$, $p_2(x),\ldots., p_n(x)$ will lead to actual solutions of this equation.

First, we shall consider the case of the series in (18). We assume that the notation for the roots $\alpha_1,\alpha_2,\ldots.,\alpha_n$ of the characteristic algebraic equation is chosen so that

$$|\alpha_1|\leqq|\alpha_2|\leqq\ldots.\leqq|\alpha_n|. \quad (20)$$

Then put

$$p_1(x)=1,\quad p_2(x)=0=p_3(x)=\ldots.=p_n(x).$$

Denote the resulting formal solution (18) by $f_1^+(x)$. Then we may write

$$
\begin{aligned}
f_1^+(x) = l_1(x) &+ S_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_1(x+n-i_1)\}\\
&+ S_x[\sum_{i_2=1}^{n}\Psi_{i_2}(x)S_{x+n-i_2}\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_1(x+n-i_1)\}]\\
&+ S_x\Big(\sum_{i_3=1}^{n}\Psi_{i_3}(x)S_{x+n-i_3}[\sum_{i_2=1}^{n}\Psi_{i_2}(x)S_{x+n-i_2}\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_1(k+n-i_1)\}]\Big)\\
&+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \qquad (21)
\end{aligned}
$$

It is convenient to introduce the term right D-region to denote a region of the complex plane defined as follows: Consider the curve made up of a semicircle C with center at the point zero and terminated at each end on the axis of imaginaries, its position being to the right of that axis and its extremities being A and B, together with the straight lines $A\infty$ and $B\infty$ drawn from the points A and B, respectively, to infinity in a direction parallel to the axis of reals and lying in the left half-plane. This curve $\infty ACB\infty$ divides the plane into two parts, one containing the point 0 and the other not containing this point. The part not containing the point 0 is called a right D-region. It is understood that this D-region does not contain the point infinity.

Now it is clear that there exists a right D-region, call it D_1, and constants M_1, M_2, M_3 such that each of the following inequalities is true so long as x is in D_1:

$$
\left.
\begin{aligned}
&|\Psi_k(x)| && < M_1|x|^{-m-2},\\
&\left|\frac{A_k(x+i+1)l_k(x)}{l_k(x+i+1)}\right| && < M_2|\alpha_k^{-i}|,\\
&|l_1(x+n-k)| && < M_3|l_1(x)|,\\
&|l_1(x+i)| && < 2|l_1(x)|\cdot|\alpha_1^i|.
\end{aligned}
\right\} k=1, 2, \dots, n. \qquad (22)
$$

It will also be assumed that D_1 is chosen so that the functions $\Psi_k(x)$, $A_k(x)$, $\{l_k(x)\}^{-1}$ are analytic in D_1.

The first term in the second member of equation (21) is a function which is analytic throughout the finite plane except at $x=0$. The second term denotes a series. It is obvious that

$$
|S_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_1(x+n-i_1)\}| \leqq \bar{S}_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_1(x+n-i_1)\},
$$

where $\bar{S}_x$ denotes the series obtained from S_x by replacing each term in the latter by its absolute value. If we confine x to the region D_1, and employ inequalities (22) and (20), we come readily to the following relations among

infinite series, these relations being valid term by term:

$$\bar{S}_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_1(x+n-i_1)\}| < \sum_{i=0}^{\infty}M_3|l_1(x+i)|\cdot nM_1|x+i|^{-m-2}\cdot M_2\sum_{k=1}^{n}|\alpha_k^{-i}|$$
$$< \sum_{i=0}^{\infty}|l_1(x)|\cdot 2n^2M_1M_2M_3|x+i|^{-m-2}. \qquad (23)$$

If we further confine x to lie in some closed region S in D_1, then it is clear that a constant $\bar{M}$ exists such that the last series above is term by term less than the series of constant terms

$$\sum_{i=0}^{\infty}\bar{M}(i+1)^{-m-2}.$$

This series being convergent, it now follows by the theorem of Weierstrass that the series denoted by

$$S_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_1(x+n-i_1)\} \qquad (24)$$

is uniformly convergent in S. Furthermore, each term of this series is analytic in S. Hence, the sum of the series is a function of x which is analytic throughout S. It is therefore analytic at every point in D_1, since S is any closed region in D_1.

By means of (23) we shall now obtain more convenient inequalities governing the character of the function in (24). We have

$$|S_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_1(x+n-i_1)\}| < 2n^2M_1M_2M_3|l_1(x)|\sum_{i=0}^{\infty}|x+i|^{-m-2}.$$

If $x=u+v\sqrt{-1}$ and u and v are real it is easy (cf. Birkhoff, *loc. cit.*, p. 248) to show that we have

$$\sum_{i=0}^{\infty}\frac{1}{|x+i|^{m+2}} < \frac{\pi}{|x|^{m+1}}, \quad |x|>1, \quad u\gtreqq 0;$$
$$\sum_{i=0}^{\infty}\frac{1}{|x+i|^{m+2}} < \frac{2\pi}{|v|^{m+1}}, \quad |v|>1.$$

Hence, if we assume that the semicircle used in defining D_1 has a radius greater than unity, we have the following inequalities valid in D_1:

$$|S_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_1(x+n-i_1)\}| < \left\{\begin{array}{ll}2\pi n^2M_1M_2M_3|l_1(x)|\cdot|x|^{-m-1}, & u\gtreqq 0,\\ 4\pi n^2M_1M_2M_3|l_1(x)|\cdot|v|^{-m-1}, & |v|>1.\end{array}\right\} \qquad (25)$$

It is obvious that the regions in which the two inequalities in (25) are separately valid overlap and that the two regions make up the whole of D_1.

We may proceed in like manner to the study of the function defined by the third term in equation (21). It is obvious that the series denoted by S_x in the expression

$$S_x[\sum_{i_2=1}^{n}\Psi_{i_2}(x)S_{x+n-i_2}\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_1(x+n-i_1)\}] \tag{26}$$

is term by term equal to or less in absolute value than the series denoted by $\bar{S}_x$ in

$$\bar{S}_x[\sum_{i_2=1}^{n}\Psi_{i_2}(x)\bar{S}_{x+n-i_2}\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_1(x+n-i_1)\}]. \tag{27}$$

It is now convenient to make a separation into two cases.

Let us consider first the case in which $u \gtreqless 0$. By employing (25) we may readily see that the series denoted by $\bar{S}_x$ in (27) is term by term less than the series

$$\bar{S}_x[\sum_{i_2=1}^{n}\Psi_{i_2}(x)\,2\pi n^2M_1M_2M_3\,|l_1(x+n-i_2)|\cdot|x+n-i_2|^{-m-1},$$

and that this in turn is term by term less than the series

$$\bar{S}_x[2\pi n^2M_1M_2M_3nM_1|x|^{-m-2}M_3|l_1(x)|\cdot|x|^{-m-1}]. \tag{28}$$

From this point we may proceed as in the discussion of the second term of the series in (21). Thus we may show that the series denoted by S_x in (26) is uniformly convergent in any closed region S lying in D_1 and having no point to the left of the axis of imaginaries, and therefore that (26) denotes a function which is analytic at every point in D_1 and not to the left of the axis of imaginaries. Moreover, through an examination of (28) it is easy to see that

$$\begin{aligned}|S_x[\sum_{i_2=1}^{n}\Psi_{i_2}(x)S_{x+n-i_2}\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_1(x+n-i_1)\}]|&\\ <(2\pi n^2M_1M_2M_3)^2|l_1(x)|\cdot|x|^{-2m-2},\quad u\gtreqless 0.&\end{aligned} \tag{29}$$

For the next discussion we confine x to D_1 and to that part of D_1 in which $|v|>1$. Then the second inequality in (27) becomes available. Making use of this and proceeding by the method just employed in the preceding paragraph, we may show that the function (26) is analytic at every point in the region in consideration and that moreover the following inequality is satisfied:

$$\begin{aligned}|S_x[\sum_{i_2=1}^{n}\Psi_{i_2}(x)S_{x+n-i_2}\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_1(x+n-i_1)\}]|&\\ <(4\pi n^2M_1M_2M_3)^2|l_1(x)|\cdot|v|^{-2m-2},\qquad |v|>1.&\end{aligned} \tag{30}$$

It is now easy to see that the method used above may be further employed in an investigation of the successive terms of the series in (21) and that the following results will thus emerge: Every term of the second member in (21) represents a function which is analytic at every point in D_1; this series is term

by term less in absolute value than each of the following series:

$$\left.\begin{array}{l} |l_1(x)|+|l_1(x)|\dfrac{M}{|x|^{m+1}}+|l_1(x)|\left(\dfrac{M}{|x|^{m+1}}\right)^2+\ldots, \\ \qquad\qquad\qquad\qquad\qquad M=2\pi n^2M_1M_2M_3, \quad u\gtreqless 0; \\ |l_1(x)|+|l_1(x)|\dfrac{2M}{|v|^{m+1}}+|l_1(x)|\left(\dfrac{2M}{|v|^{m+1}}\right)^2+\ldots, \quad |v|>1, \end{array}\right\} \tag{31}$$

each inequality being valid for that part of D_1 for which the corresponding adjoined condition $u\gtreqless 0$ or $|v|>1$ is satisfied.

Let us put on the region D_1 a further restriction consistent with those which we have previously placed upon it, namely, that it shall have the property that for any x in D_1, where $x=u+v\sqrt{-1}$, the following two inequalities shall be true:

$$M|x|^{-m-1}<1 \text{ when } u\gtreqless 0; \quad 2M|v|^{-m-1}<1 \text{ when } |v|>\beta,$$

β being an appropriately chosen constant. Now let S be any closed region lying in the modified D_1. Then, from the result associated with (31) above, we see readily that the series in (21) is uniformly convergent in S; whence it follows that the sum $f_1^+(x)$ is analytic at every point in the region D_1, since the terms of that series are analytic at all such points. Moreover, it is clear that $f_1^+(x)$ satisfies the following inequalities, valid in D_1:

$$\left.\begin{array}{ll} |f_1^+(x)-l_1(x)|<|l_1(x)|\dfrac{M}{|x|^{m+1}-M}, & u\gtreqless 0; \\ |f_1^+(x)-l_1(x)|<|l_1(x)|\dfrac{2M}{|v|^{m+1}-2M}, & |v|>\beta, \end{array}\right\} \tag{31'}$$

as one sees readily by transposing the first term in the series in (21) to the first member of the equation and comparing the resulting second member term by term with the series in (31), exclusive of the first term in each, and finally summing the last series (with first term removed).

From the last inequality we have readily the following limits:

$$\left.\begin{array}{l} \lim\limits_{x=\infty} x^m\left\{\dfrac{f_1^+(x)}{l_1(x)}-1\right\}=0, \\ \lim\limits_{v=\pm\infty} v^m\left\{\dfrac{f_1^+(x)}{l_1(x)}-1\right\}=0, \end{array}\right\} \tag{32}$$

the first being valid for any approach of x to infinity in the right half-plane, and the second being valid for the approach of the real variable v to either $+\infty$ or $-\infty$. Since $v=|x|\sin(\arg x)$, it is easy to see from these two limits that

$$\lim_{x=\infty} x^m\left\{\frac{f_1^+(x)}{l_1(x)}-1\right\}=0, \tag{33}$$

provided that x approaches infinity in any way so as to remain always without any sector determined by two rays from zero to infinity and including between them the negative part of the axis of reals.

By actual substitution it may now readily be shown that the function $f_1^+(x)$ satisfies equation (2) for all x in D_1. By means of the equation

$$f(x)=-\frac{1}{a_n(x)}f(x+n)-\frac{a_1(x)}{a_n(x)}f(x+n-1)-\ldots-\frac{a_{n-1}(x)}{a_n(x)}f(x+1),$$

which is obtained from (2) in an obvious manner, one may further define the solution $f_1^+(x)$ in the part of the finite plane exterior to D_1. It may thus be seen that the extended solution $f_1^+(x)$ is analytic throughout the finite plane except at the singularities of the functions

$$\frac{1}{a_n(x)},\quad \frac{a_1(x)}{a_n(x)},\ \ldots,\ \frac{a_{n-1}(x)}{a_n(x)} \tag{34}$$

and points congruent to them on the left. (A point $\alpha-i$ is said to be congruent to α on the left, if i is a positive integer; similarly, we say that $\alpha+i$ is congruent to α on the right, if i is a positive integer.) It is further obvious that the singularities of $f_1^+(x)$ are poles in case the functions (34) are rational and that the complete set of numbers each of which is the order of a pole of $f_1^+(x)$ is a bounded set.

In obtaining the solution whose properties we have just developed, we began with the series (21) which at first was only known to have formal validity and by means of it arrived at the actual solution $f_1^+(x)$. Similarly, one might start from the formal solution

$$\begin{aligned} f_n^-(x)=l_n(x)&+T_x\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_n(x+n-i_1)\}\\ &+T_x[\sum_{i_2=1}^{n}\Psi_{i_2}(x)T_{x+n-i_2}\{\sum_{i_1=1}^{n}\Psi_{i_1}(x)l_n(x+n-i_1)\}]+\ldots, \end{aligned}\tag{35}$$

obtained from (19) in a way similar to that by which (21) was obtained from (18), and show that we are thus led to an actual solution $f_n^-(x)$ of (9). There would be no essential modification in the style of the argument. Instead of a right D-region we would employ a left D-region, such a region being defined by saying that its points are obtained from those of a right D-region by reflection through the axis of imaginaries. In fact, practically the only modification of the argument necessary is that which comes of an interchange of the rôles of the right and left sides of the plane.

Combining the principal conclusions which we reached in the treatment of equation (21) and those which will emerge from a discussion of (35) in the manner indicated, we have the following result:

Preliminary Theorem. *If the characteristic algebraic equation associated with the difference equation* (2) *has its roots* $\alpha_1, \alpha_2, \ldots, \alpha_n$ *different from each other and from zero and these are ordered so that* $|\alpha_1| \leqq |\alpha_2| \leqq \ldots \leqq |\alpha_n|$, *then this difference equation has two particular solutions* $f_1^+(x)$ *and* $f_n^-(x)$, *possessing the following properties:*

1) *The solution* $f_1^+(x)$ *is analytic throughout the finite plane except at the singularities of the functions* (34) *and points congruent to them on the left. The solution* $f_n^-(x)$ *is analytic throughout the finite plane except at the points congruent on the right* (*at a distance of n units or more*) *to the singularities of the functions* $a_1(x), a_2(x), \ldots, a_n(x)$. *Moreover, the singularities of each solution in the finite plane are poles provided that* $a_1(x), a_2(x), \ldots, a_n(x)$ *are rational functions, and the complete set of numbers each of which is the order of one of these poles is a bounded set.*

2) *There exists a right D-region RD* [*left D-region LD*] *in which the solution* $f_1^+(x)$ [$f_n^-(x)$] *is representable by the series* (21) [(35)], *this series being absolutely and uniformly convergent throughout any closed region S lying entirely in RD* [*LD*]. *Equation* (2) *itself then suffices to determine the solution in that part of the plane not in the region RD* [*LD*].

3) *If x approaches infinity in any way so as to remain always without any sector* (*however small*) *determined by two rays from zero to infinity and including between them the negative* [*positive*] *part of the axis of reals, then*

$$\lim x^m \left\{ \frac{f_1^+(x)}{l_1(x)} - 1 \right\} = 0 \qquad \left[\lim x^m \left\{ \frac{f_n^-(x)}{l_n(x)} - 1 \right\} = 0 \right],$$

$l_1(x)$ [$l_n(x)$] *being the function denoted by that symbol in the first equation in* § 1. *Also,*

$$\lim_{v=\pm\infty} v^m \left\{ \frac{f_1^+(x)}{l_1(x)} - 1 \right\} = 0 \qquad \left[\lim_{v=\pm\infty} v^m \left\{ \frac{f_n^-(x)}{l_n(x)} - 1 \right\} = 0 \right], \quad x = u + v\sqrt{-1}.$$

Remark. It is important for later use to point out the fact that, in so far as any statement in the conclusion of the preliminary theorem refers to a region RD [LD] or any part of it, it is independent of the nature of the coefficients $\Psi_1(x), \Psi_2(x), \ldots, \Psi_n(x)$ of (9), and hence of the coefficients $a_1(x), a_2(x), \ldots,$ $a_n(x)$ of (2), in the part of the plane exterior to RD [LD]. More precisely, *if the functions* $\Psi_1(x), \Psi_2(x), \ldots, \Psi_n(x)$ *are analytic in some right* [*left*] *D-region and throughout such a region are represented asymptotically by the expansions in* (10) *or throughout such a region merely satisfy an inequality of the form*

$$|\Psi_k(x)| < \frac{\bar{\mu}}{|x|^{m+2}},$$

where $\bar{\mu}$ is a constant, then there exists a solution $f_1^+(x)$ [$f_n^-(x)$] and a right [left] D-region RD [LD] such that $f_1^+(x)$ [$f_n^-(x)$] has in RD [LD] the same properties as the solution $f_1^+(x)$ [$f_n^-(x)$] given in the theorem. Likewise the inequalities derived above for such solutions in such D-regions remain valid under the less restrictive hypotheses.

It should be observed, at this point, that the two solutions obtained above were gotten by the aid of certain functions $l_1(x), l_2(x), \ldots, l_n(x)$, these being obtained from the formal expansions (5) by breaking them off at the $(m+1)$-th term. It follows therefore that the functions $f_1^+(x)$ and $f_n^-(x)$ are now to be treated as functions of m as well as of x; or, more exactly, we are to look upon them as possibly functions of m. Ultimately we shall show that they are indeed independent of m (at least when m is sufficiently large), this proof being readily made after we have obtained a fundamental set of solutions, each solution being a possible function of m. As soon as this independence of m is proved, we shall be able to obtain readily the asymptotic character of $f_1^+(x)$ and $f_n^-(x)$. It is clear that the limits in the above theorem are sufficient for this.

§ 3. *The First System of Intermediate Solutions.*

For the difference equation (2) we shall now obtain four sets of n particular solutions each, the solutions in each set being independent with respect to periodic multipliers of period 1, and therefore constituting a fundamental system of solutions.

We shall first start from the particular solution $f_1^+(x)$ obtained in the preceding section. By means of the substitution

$$f(x) = f_1^+(x)\,g(x), \tag{36}$$

equation (2) goes over into the form

$$g(x+n) + k a_1(x) f_1^+(x+n-1) g(x+n-1) + \ldots + k a_n(x) f_1^+(x) g(x) = 0, \tag{37}$$

where $k = 1/f_1^+(x+n)$. If we put

$$G(x) = \Delta g(x),$$

the preceding equation becomes

$$G(x+n-1) + B_1(x)\,G(x+n-2) + \ldots + B_{n-1}(x)\,G(x) = 0, \tag{38}$$

where

$$\begin{aligned}
B_1(x) &= 1 + k a_1(x)\,f_1^+(x+n-1),\\
B_2(x) &= 1 + k a_1(x)\,f_1^+(x+n-1) + k a_2(x)\,f_1^+(x+n-2),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
B_{n-1}(x) &= 1 + k a_1(x)\,f_1^+(x+n-1) + \ldots + k a_{n-1}(x)\,f_1^+(x+1).
\end{aligned}$$

From the way in which equation (37) was found, it is clear that its formal power series solutions $g_i(x)$ may be determined from the relations

$$g_i(x) = \frac{a_i^x x^{\mu_i}(1+c_{i1}x^{-1}+c_{i2}x^{-2}+\ldots.)}{a_1^x x^{\mu_1}(1+c_{11}x^{-1}+c_{12}x^{-2}+\ldots.)}, \quad i=1, 2, \ldots., n, \tag{39}$$

by formal division in the second member for each value of i. Furthermore, the formal power series solutions $G_i(x)$ of (38) may be obtained from these by reckoning out formally the differences $\Delta g_i(x)$, $i=1, 2, \ldots., n$. Thus we have

$$G_i(x) = \left(\frac{\alpha_{i+1}}{\alpha_1}\right)^x x^{\mu_{i+1}-\mu_1}\left(\frac{\alpha_{i+1}}{\alpha_1}-1\right)\left(1+\frac{d_{i1}}{x}+\frac{d_{i2}}{x^2}+\ldots.\right), \; i=1, 2, \ldots., n-1. \tag{40}$$

Let us denote by $L_i(x)$, $i=1, 2, \ldots., n-1$, the functions

$$L_i(x) = \left(\frac{\alpha_{i+1}}{\alpha_1}\right)^x x^{\mu_{i+1}-\mu_1}\left(\frac{\alpha_{i+1}}{\alpha_1}-1\right)\left(1+\frac{d_{i1}}{x}+\ldots.+\frac{d_{im}}{x^m}\right), \; i=1, 2, \ldots., n-1, \tag{41}$$

where m has the same value as in the preceding section.

An examination of equation (38) with reference to the first remark following the theorem of the preceding section is sufficient to bring out the fact that there exists a right D-region in which the coefficients $B_1(x), \ldots.,$ $B_{n-1}(x)$ satisfy the conditions requisite for the application of that part of the theorem which relates to the right D-regions. In order to set up in detail the inequalities by which this may be proved, it is necessary to separate equation (38) into two members after the manner in which equation (2) was separated into the two members in (9), and then to employ the inequalities by which we have found $f_1^+(x)$ to be restricted. There is nothing involved in this more than the straightforward algebra of inequalities, and one can even see beforehand what must be the result of the computation, so that it is unnecessary to perform it.

By means of the theorem of the preceding section we now have the following result: There exists a right D-region RD and a solution $G_1^+(x)$ of equation (38) such that $G_1^+(x)$ is analytic throughout RD, and is representable in such a region by a series of type (21) which is absolutely and uniformly convergent throughout any closed region S lying entirely in RD. If x approaches infinity in any way so as to remain always without any sector (however small) determined by two rays from 0 to ∞ and including between them the negative part of the axis of reals, then

$$\lim x^m \left\{\frac{G_1^+(x)}{L_1(x)}-1\right\} = 0.$$

Also,

$$\lim_{v=\pm\infty} v^m \left\{\frac{G_1^+(x)}{L_1(x)}-1\right\} = 0.$$

These limits may be replaced by the somewhat stronger inequalities by which they were derived, namely, those corresponding to (31′). Then for appropriately chosen constants K and β we have

$$\left.\begin{aligned} |G_1^+(x)-L_1(x)| &< |L_1(x)|\cdot K|x|^{-m-1}, \quad u \gtreqless 0, \\ |G_1^+(x)-L_1(x)| &< |L_1(x)|\cdot K|v|^{-m-1}, \quad |v|>\beta, \end{aligned}\right\} \tag{42}$$

where $x=u+v\sqrt{-1}$ and lies in the region RD.

Now, equation (37) has particular solutions $g_1^+(x)$ such that

$$\Delta g_1^+(x)=G_1^+(x). \tag{43}$$

There are two solutions of (43) from each of which we are led to interesting particular solutions of (37) and hence of (2). Of these we shall treat one in detail in this section and the other in a later section.

Equation (43) has the obvious formal solution

$$\bar{g}_1^+(x)=G_1^+(x-1)+G_1^+(x-2)+G_1^+(x-3)+\ldots.. \tag{44}$$

In order to treat conveniently the matter of convergence of this series, it will be necessary to put on the further restriction that $|\alpha_1|<|\alpha_2|$. (Compare equation (20).) Then, from the second inequality in (42), it follows readily that the series in equation (44) converges for every value of x such that $x-1$' $x-2$, $x-3$, ... all lie in RD; that is, it converges for every x such that $|v|\geqq R_1$, where R_1 is the radius of the semicircle by means of which the region RD is defined. Moreover, if S is any closed region throughout which $|v|\geqq R_1$, then the series in (44) is clearly uniformly convergent in S. But each term of this series is analytic in S. Hence, $\bar{g}_1^+(x)$ is analytic in S; it is therefore analytic at every point of the finite plane for which $|v|\geqq R_1$.

We may also obtain an upper bound to the absolute value of $\bar{g}_1^+(x)$ when $|v|\geqq R_1$. We observe that R_1 should be chosen greater than $2K$ and greater than β so as to make valid all our inequalities. Define a function $S(x)$ by means of the series

$$S(x)=L_1(x-1)+L_1(x-2)+L_1(x-3)+\ldots.. \tag{45}$$

It is easy to see that this series is absolutely and uniformly convergent when x is confined to a closed region S throughout which $|v|\geqq R_1$. Then, from (42), (44) and (45) we have

$$|\bar{g}_1^+(x)-S(x)|<\sum_{i=1}^{\infty}|L_1(x-i)|\cdot K|v|^{-m-1}, \quad |v|\geqq R_1.$$

Now, positive constants N_1 and N_2 exist such that

$$N_1\left|\frac{\alpha_2}{a_1}\right|^{-i}<\frac{|L_1(x-i)|}{|L_1(x)|}<N_2\left|\frac{\alpha_2}{\alpha_1}\right|^{-i},$$

provided that $|v|$ is sufficiently large. Hence,

$$|\bar{g}_1^+(x)-S(x)|<N_2K|L_1(x)|\frac{1}{|v|^{m+1}}\sum_{i=1}^{\infty}\left|\frac{\alpha_2}{\alpha_1}\right|^{-i}.$$

From (45) we see that $|S(x)|/|L_1(x)|$ is bounded away from zero.* From this fact and the last relation it follows readily that a positive constant N_3 exists such that

$$\left|\frac{\bar{g}_1^+(x)}{S(x)}-1\right|<\frac{N_3}{|v|^{m+1}}. \tag{46}$$

Referring to the definitions of $L_1(x)$, $l_1(x)$ and $l_2(x)$, we see that

$$L_1(x)=\frac{l_2(x+1)}{l_1(x+1)}-\frac{l_2(x)}{l_1(x)}+\left(\frac{\alpha_2}{\alpha_1}\right)^x x^{\mu_2-\mu_1}\left(\frac{\gamma_{m+1}}{x^{m+1}}+\frac{\gamma_{m+2}}{x^{m+2}}+\ldots.\right), \tag{46a}$$

where $\gamma_{m+1},\ldots.$ are constants, the expansion being valid for sufficiently large values of $|x|$. From this relation and (45) it follows that

$$S(x)=\frac{l_2(x)}{l_1(x)}+\sum_{i=1}^{\infty}\left(\frac{\alpha_2}{\alpha_1}\right)^{x-i}(x-i)^{\mu_2-\mu_1}\left(\frac{\gamma_{m+1}}{(x-i)^{m+1}}+\ldots.\right). \tag{45'}$$

On transposing the term $l_2(x)/l_1(x)$ and dividing the resulting equation through by this term, we have a relation from which it is easy to show that a positive constant N_4 exists such that

$$\left|\frac{S(x)l_1(x)}{l_2(x)}-1\right|<\sum_{i=1}^{\infty}\left|\frac{\alpha_2}{\alpha_1}\right|^{-i}\frac{N_4}{|x-i|^{m+1}}<\frac{N_4}{|v|^{m+1}}\sum_{i=1}^{\infty}\left|\frac{\alpha_2}{\alpha_1}\right|^{-i}.$$

In showing this, it is necessary to observe that

$$\frac{l_2(x)}{l_1(x)}=\left(\frac{\alpha_2}{\alpha_1}\right)^x x^{\mu_2-\mu_1}(1+\ldots.),$$

the quantity in parenthesis being a descending power series in x. Hence, a positive constant N_5 exists such that

$$\left|\frac{S(x)l_1(x)}{l_2(x)}-1\right|<\frac{N_5}{|v|^{m+1}}. \tag{47}$$

Multiply relations (46) and (47) together member by member and combine the resulting inequality with (46) and (47); thus we have

$$\left|\frac{\bar{g}_1^+(x)l_1(x)}{l_2(x)}-1\right|<\frac{N_6}{|v|^{m+1}}, \tag{48}$$

where N_6 is a positive constant.

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* Or this may more readily be proved by aid of relation (47) and the fact that $L_1(x)l_1(x)/l_2(x)$ approaches $(a_2/a_1)-1$ as x approaches infinity, the latter fact being readily demonstrated by aid of equation (46a).

Now, a solution $\bar{f}_2^+(x)$ of equation (2) is given by the relation

$$\bar{f}_2^+(x) = f_1^+(x)\bar{g}_1^+(x),$$

as one sees from (36). This solution is analytic for $|v|$ sufficiently large. From (31′) we have a relation of the form

$$\left|\frac{f_1^+(x)}{l_1(x)} - 1\right| < \frac{N_7}{|v|^{m+1}},$$

where N_7 is a positive constant. From this relation and (48) it follows readily that a constant N exists such that

$$\left|\frac{\bar{f}_2^+(x)}{l_2(x)} - 1\right| < \frac{N}{|v|^{m+1}}, \quad |v| \geqq \bar{R}, \tag{49}$$

where $\bar{R}$ is a positive constant. This is one of the fundamental inequalities for $\bar{f}_2^+(x)$.

A similar one may be found corresponding to the first inequality in (42). It may be written

$$\left|\frac{\bar{f}_2^+(x)}{l_2(x)} - 1\right| < \frac{\mu}{|x|^{m+1}}, \quad u \geqq 0, \quad |v| \geqq \bar{R}, \tag{50}$$

where μ is a positive constant. In view of (49) it is obviously sufficient to prove this for the case when x is further limited to lie in a sector made by two rays from zero to infinity and lying entirely in the right half-plane. Furthermore, it is clear that $|x|$, and hence u, may be taken as large as one pleases.

We start from equation (44). Choose x so that u is large and denote by k the greatest integer such that $2k \leqq u$. Write (44) in the form

$$\bar{g}_1^+(x) = \sum_{i=1}^{k} G_1^+(x-i) + \sum_{i=k+1}^{\infty} G_1^+(x-i).$$

Then

$$\bar{g}_1^+(x) - S(x) = \sum_{i=1}^{k} \{G_1^+(x-i) - L_1(x-i)\} + \sum_{i=k+1}^{\infty} \{G_1^+(x-i) - L_1(x-i)\},$$

so that, in view of (42), we have

$$|\bar{g}_1^+(x) - S(x)| < \sum_{i=1}^{k} |L_1(x-i)| \frac{K}{|x-i|^{m+1}} + \sum_{i=k+1}^{\infty} |L_1(x-i)| \frac{K}{|v|^{m+1}}, \tag{51}$$

since it is clear that the first inequality in (42) is applicable to the first summation and the second inequality to the second.

Now, it is clear that a positive constant μ_1 exists such that

$$\left|\frac{L_1(x-i)}{L_1(x)}\right| < \mu_1 \left|\frac{\alpha_2}{\alpha_1}\right|^{-i}, \quad i = 1, 2, \ldots, k.$$

Then from (51) we have

$$|\bar{g}_1^+(x)-S(x)| < \mu_1 |L_1(x)| \frac{2K}{|x|^{m+1}} \sum_{i=1}^{k} \left|\frac{\alpha_2}{\alpha_1}\right|^{-i} + \mu_1 |L_1(x)| \frac{K}{|v|^{m+1}} \sum_{i=k+1}^{\infty} \left|\frac{\alpha_2}{\alpha_1}\right|^{-i}.$$

If $|x|$ increases indefinitely, it is clear that the second term in the second member becomes indefinitely small with respect to the first term; to see this, it is sufficient to observe the rôle of the exponential quantity $|\alpha_2/\alpha_1|^{-k}$. Therefore, a positive constant μ_2 exists such that

$$|\bar{g}_1^+(x)-S(x)| < |L_1(x)| \cdot \mu_2 |x|^{-m-1}.$$

From this point forward the march of the argument is like that in the preceding treatment. As in the previous case it may readily be shown that $|S(x)|/|L_1(x)|$ is bounded away from zero; hence we see from the last inequality that a positive constant μ_3 exists such that

$$\left|\frac{\bar{g}_1^+(x)}{S(x)}-1\right| < \mu_3 |x|^{-m-1}.$$

By a method similar to that by which (47) was proved, it may be shown that a positive constant μ_4 exists such that

$$\left|\frac{S(x)\, l_1(x)}{l_2(x)}-1\right| < \mu_4 |x|^{-m-1}.$$

In the demonstration of this result it will be necessary to separate the series for $S(x)$ in (45′) into two series, one of k terms and the other of the remaining terms. The argument is similar to that just employed; it may readily be supplied by the reader.

Making use of the last two relations and continuing precisely as in the argument following (47) one may readily complete the proof of (50). Thus, we have the requisite properties of the second solution $\bar{f}_2^+(x)$ of equation (2) in case $|\alpha_1| < |\alpha_2|$.

In case $|\alpha_1| = |\alpha_2|$ the method by which $f_1^+(x)$ was found is obviously valid for a second solution $f_2^+(x)$. We now call it $\bar{f}_2^+(x)$. It has the properties expressed in (49) and (50).

Now, apply to equation (38) the result which we have just derived for equation (2). We see that (38) has a second solution $G_2^+(x)$ such that

$$\begin{aligned} |G_2^+(x)-L_2(x)| &< |L_2(x)| \cdot K_1 |x|^{-m-1}, \quad u \gtreqless 0; \\ |G_2^+(x)-L_2(x)| &< |L_2(x)| \cdot K_1 |v|^{-m-1}, \end{aligned}$$

where K_1 is a positive constant, these relations being valid when $|v|$ is sufficiently large. If we start from this solution and proceed by a method similar to that employed in the discussion associated with $G_1^+(x)$, it is clear that we shall be led to a third solution $\bar{f}_3^+(x)$ of equation (2), that this solution will be

analytic for $|v|$ sufficiently large and that it will verify the following relations:

$$\left.\begin{aligned} &\left|\frac{\bar{f}_3^+(x)}{l_3(x)}-1\right| < \frac{\mu'}{|x|^{m+1}}, \quad u \gtreqless 0; \\ &\left|\frac{\bar{f}_3^+(x)}{l_3(x)}-1\right| < \frac{\mu'}{|v|^{m+1}}, \end{aligned}\right\} \tag{52}$$

when $|v|$ is sufficiently large, μ' being a constant.

By continuing this process of interaction between equations (2) and (38), we shall be led finally to a set of n solutions,

$$\bar{f}_1^+(x) = f_1^+(x),\ \bar{f}_2^+(x),\ \ldots,\ \bar{f}_n^+(x), \tag{53}$$

of equation (2), each solution being analytic when $|v|$ is sufficiently large. From the inequalities such as (52), by which these functions are bounded, we have the following relations:

$$\left.\begin{aligned} &\lim x^m \left\{\frac{\bar{f}_i^+(x)}{l_i(x)}-1\right\} = 0, \quad i=1, 2, \ldots, n, \\ &\lim_{v=\pm\infty} v^m \left\{\frac{\bar{f}_i^+(x)}{l_i(x)}-1\right\} = 0, \quad i=1, 2, \ldots, n, \end{aligned}\right\} \tag{54}$$

the first relation being valid for x approaching infinity in any way so as to satisfy the two conditions that $|v|$ is sufficiently large and that x remains without a sector formed by two rays from zero to infinity and including between them the negative part of the axis of reals.

That these solutions are linearly independent with respect to periodic multipliers of period 1, and hence constitute a fundamental set of solutions of (2), is easily shown. It is sufficient to prove that the determinant

$$D(x) \equiv \begin{vmatrix} \bar{f}_1^+(x) & \bar{f}_2^+(x) & \ldots \bar{f}_n^+(x) \\ \bar{f}_1^+(x+1) & \bar{f}_2^+(x+1) & \ldots \bar{f}_n^+(x+1) \\ \ldots & \ldots & \ldots \\ \bar{f}_1^+(x+n-1) & \bar{f}_2^+(x+n-1) & \ldots \bar{f}_n^+(x+n-1) \end{vmatrix}$$

is not identically equal to zero. Divide the k-th column of this determinant by $\alpha_k^x x^{\mu_k}$, $k=1, 2, \ldots, n$. It is sufficient to prove that the resulting determinant is not identically zero. But in view of (54) it may be seen that this determinant approaches the value $\bar{D}(\infty)$ as $|v|$ approaches infinity, $\bar{D}(\infty)$ having the same meaning as in the first part of § 1. This is different from zero, since $\alpha_1, \ldots, \alpha_n$ are all different. Therefore, the set (53) is a fundamental system of solutions of equation (2).

We have seen that the function $\bar{f}_1^+(x)$ was possibly dependent on m. The other functions in (53) were obtained by aid of $\bar{f}_1^+(x)$, and therefore they are

possibly functions of m. We shall now show that they are indeed independent of m, at least if m is sufficiently large.

More generally, let $f_k(x)$ be any solution of equation (2) such that

$$\lim x^m \left\{ \frac{f_k(x)}{l_k(x)} - 1 \right\} = 0 \tag{55}$$

when $x, = u + v\sqrt{-1}$, approaches infinity in such a way that u approaches $+\infty$, and $|v|$ becomes sufficiently large, say $|v| \gtreqless V$.

Since $f_k(x)$ is a solution of equation (2) it may be written in the form

$$f_k(x) = p_1(x)\bar{f}_1^+(x) + p_2(x)\bar{f}_2^+(x) + \ldots + p_n(x)\bar{f}_n^+(x), \tag{56}$$

where $p_1(x)$, $p_2(x)$, $\ldots$, $p_n(x)$ are suitably determined periodic functions of x of period 1. Then, from (55) and the first equation in (54), we see that

$$\lim x^m \left\{ p_1(x) \frac{l_1(x)}{l_k(x)} + p_2(x) \frac{l_2(x)}{l_k(x)} + \ldots + p_n(x) \frac{l_n(x)}{l_k(x)} - 1 \right\} = 0.$$

Let $x_0, = u_0 + v_0\sqrt{-1}$, be any value of x such that $|v_0| \geqq V$, let t be a variable running over the set 0, 1, 2, 3, $\ldots$, and let s be a non-negative integer. Then on replacing x by $x_0 + t + s$ in the preceding limit and reducing, we have the relations

$$\lim_{t=\infty} t^m \{ r_{k1} u_{k1}^t t^{\mu_1 - \mu_k} \cdot u_{k1}^s (1 + \varepsilon_{1s}) + \ldots + r_{kn} u_{kn}^t t^{\mu_n - \mu_k} \cdot u_{kn}^s (1 + \varepsilon_{ns}) \} = 0,$$

where $u_{kl} = \alpha_l / \alpha_k$, ε_{ls} approaches zero as t approaches infinity, and

$$r_{kl} = p_l(x_0) u_{kl}^{x_0}, \qquad l \neq k, \quad r_{kk} = p_k(x_0) - 1.$$

It is to be observed that no two of the quantities u_{kl} are equal and that each of them is different from 0.

Now suppose that m is not less than the real part of any of the differences $\mu_1 - \mu_k$, $\mu_2 - \mu_k$, $\ldots$, $\mu_n - \mu_k$. Then from the preceding limit we have the following:

$$\lim_{t=\infty} \{ r_{k1} u_{k1}^t t^{v_1} \cdot u_{k1}^s (1 + \varepsilon_{1s}) + \ldots + r_{kn} u_{kn}^t t^{v_n} \cdot u_{kn}^s (1 + \varepsilon_{ns}) \} = 0, \tag{57}$$

where the real part of v_l is zero or positive for every value of l. This relation is valid for any non-negative integral value of s.

Consider the set of relations obtained from (57) by giving to s the values $s = 0, 1, 2, \ldots, n-1$. By multiplying these in order by the cofactors (in order) of the elements in the l-th column of the determinant

$$\Delta = \begin{vmatrix} 1 + \varepsilon_{10} & 1 + \varepsilon_{20} & \ldots . 1 + \varepsilon_{n0} \\ u_{k1}(1 + \varepsilon_{11}) & u_{k2}(1 + \varepsilon_{21}) & \ldots . u_{kn}(1 + \varepsilon_{n1}) \\ \ldots & \ldots & \ldots \\ u_{k1}^{n-1}(1 + \varepsilon_{1,n-1}) & \ldots & \ldots \end{vmatrix}$$

and adding the resulting relations, we have

$$\lim_{t=\infty} r_{kl} u_{kl}^t t^{v_l} \Delta = 0.$$

Since

$$\lim_{t=\infty} \Delta = \begin{vmatrix} 1 & 1 & \ldots\ldots 1 \\ u_{k1} & u_{k2} & \ldots\ldots u_{kn} \\ u_{k1}^2 & u_{k2}^2 & \ldots\ldots u_{kn}^2 \\ \ldots & \ldots & \ldots\ldots \\ u_{k1}^{n-1} & u_{k2}^{n-1} & \ldots\ldots u_{kn}^{n-1} \end{vmatrix} \neq 0,$$

we have

$$\lim_{t=\infty} r_{kl} u_{kl}^t t^{v_l} = 0.$$

This relation can be true only if $|u_{kl}| < 1$ or $r_{kl} = 0$, since the real part of v_l is not less than 0. But $|u_{kl}| = |\alpha_l / \alpha_k| \geqq 1$ unless $l < k$. Therefore $r_{kl} = 0$ if $l \geqq k$. Hence, $p_k(x_0) = 1$ and $p_l(x_0) = 0$ if $l > k$. Therefore, we see from (56) that

$$f_k(x) = p_1(x) \bar{f}_1^+(x) + \ldots\ldots + p_{k-1}(x) \bar{f}_{k-1}^+(x) + \bar{f}_k^+(x), \qquad |v| \geqq V, \quad (58)$$

where $p_1(x), \ldots\ldots, p_{k-1}(x)$ are suitably determined periodic functions.

From the particular case when $k=1$ we see that a solution $f_1(x)$ of equation (2) which satisfies the relation (55) for $k=1$ coincides with $\bar{f}_1^+(x)$, at least if m is greater than the real part of each of the quantities $\mu_2 - \mu_1$, $\mu_3 - \mu_1$, $\ldots\ldots, \mu_n - \mu_1$. From this it follows readily that each of the solutions of (2) in the set (53) is independent of m, provided that m is greater than the real part of every one of the differences $\mu_i - \mu_j$, $j = 1, 2, \ldots\ldots, n$, $i = j+1, \ldots\ldots, n$. In order to see this, it is only necessary to observe the manner in which these solutions were obtained.

It should be observed that the only use made of the fact that the real part of v_l is not less than zero was to insure that $u_{kl}^t t^{v_l}$ does not approach zero, for t becoming infinite, when $l \geqq k$. This condition is obviously satisfied without regard to the value of v_l, provided that the roots $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n$ of the characteristic algebraic equation are different in absolute value, for then $|u_{kl}| > 1$ when $l > k$. Hence we may take $m = 0$ for the case when the α's are all different in absolute value.

We shall now determine largely the asymptotic character of the solutions (53) under the hypothesis that m is greater than the real part of each of the differences $\mu_i - \mu_j$, $j = 1, 2, \ldots\ldots, n$, $i = j+1, \ldots\ldots, n$, or without restriction on m when the roots of the characteristic algebraic equation are different in absolute value.

In this connection it is convenient to distinguish (cf. Birkhoff, *loc. cit.*, p. 248) between two kinds of asymptotic representation of a function by a series of the form

$$S(x)=x^{sx}\alpha^x x^\mu\left(c_0+\frac{c_1}{x}+\frac{c_2}{x^2}+\ldots.\right).$$

Let $g(x)$ be the given function; if for each m the difference

$$d(x)=g(x)x^{-sx}\alpha^{-x}x^{-\mu}-\left(c_0+\frac{c_1}{x}+\ldots.+\frac{c_m}{x^m}\right)$$

becomes uniformly small of order m; that is, if $\lim x^m d(x)=0$ for x approaching infinity in a certain region, we say that $g(x)$ is asymptotically represented by $S(x)$ in that region, *with respect to* x; if, on the other hand, we have $\lim\limits_{v=\pm\infty} v^m d(x)=0$, we say that $g(x)$ is asymptotically represented by $S(x)$ in that region, *with respect to* v.

Making use of relation (54) for increasingly large values of m and remembering that $\overline{f}_i^+(x)$ is independent of m, we see readily that $\overline{f}_i^+(x)$ is asymptotic to

$$\alpha_i^x x^{\mu_i}\left(1+\frac{c_{i1}}{x}+\frac{c_{i2}}{x^2}+\ldots.\right)$$

with respect to v and also with respect to x, provided that x approaches infinity so as to satisfy the conditions that $|v|$ is sufficiently large and that x remains without a sector formed by two rays from zero to infinity and including between them the negative part of the axis of reals.

The fundamental properties of the functions $\overline{f}_1^+(x),\ldots.,\overline{f}_n^+(x)$ are now determined throughout a region in which $|v|$ is sufficiently great, say $|v|\geqq V$, and x is to the right of some line parallel to the axis of imaginaries. Their properties in the remainder of that part of the plane for which $|v|\geq V$ may be determined, as in the case of $f_1^+(x)$ in § 2, by means of equation (2) itself.

The principal results indicated in this section may be put together in the form of the following theorem:

First System of Intermediate Solutions.* *If the characteristic algebraic equation associated with the difference equation* (2) *has its roots different from each other and from zero, then the difference equation has the fundamental system of solutions*

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* The existence of these solutions was first pointed out in my memoir, *loc. cit.*, p. 119. Their properties were first developed in Birkhoff's memoir, already referred to.

$$\bar{f}_1^+(x),\ \bar{f}_2^+(x),\ \ldots,\ \bar{f}_n^+(x), \tag{53 bis}$$

determined above, and these solutions have the following properties:

1) *A constant V exists such that each function in the set is analytic when $|v| \geqq V$, where v is defined by writing $x=u+v\sqrt{-1}$, u and v being real;*

2) *The asymptotic relations*

$$\bar{f}_i^+(x) \sim \alpha_i^x x^{\mu_i}\left(1+\frac{c_{i1}}{x}+\frac{c_{i2}}{x^2}+\ldots\right), \qquad i=1,\,2,\,\ldots,\,n,$$

exist and are valid with respect to x in that part of any right half-plane for which $|v| \geqq V$ and with respect to v in the entire plane.*

(By a right [left] half-plane is meant that part of the plane which is to the right [left] of a line parallel to the axis of imaginaries.)

Let us retain for m the value 0 if the roots of the characteristic equation are different in absolute value and in the contrary case a value not less than the real part of any of the differences $\mu_i-\mu_j$, $j=1,\,2,\,\ldots,\,n$, $i=j+1,\,\ldots,\,n$. Denote by $l_i(x)$, $i=1,\,2,\,\ldots,\,n$, the functions so represented in the first part of § 1. Then a solution $f(x)$ of equation (2),

$$f(x)=p_1(x)\bar{f}_1^+(x)+p_2(x)\bar{f}_2^+(x)+\ldots+p_n(x)\bar{f}_n^+(x), \tag{59}$$

where $p_1(x),\ p_2(x),\ \ldots,\ p_n(x)$ are periodic functions of x of period 1, has an important set of properties obtainable from the first relation in (54), namely,

$$\left.\begin{aligned}
&\lim_{t=\infty} t^m\left\{\frac{f(x_0+t)}{l_n(x_0+t)}-p_n(x_0)\right\}=0,\\
&\lim_{t=\infty} t^m\left\{\frac{f(x_0+t)-p_n(x_0)\bar{f}_n^+(x_0+t)}{l_{n-1}(x_0+t)}-p_{n-1}(x_0)\right\}=0, \qquad |v| \geqq V,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\lim_{t=\infty} t^m\left\{\frac{f(x_0+t)-p_n(x_0)\bar{f}_n^+(x_0+t)-\ldots-p_2(x_0)\bar{f}_2^+(x_0+t)}{l_1(x_0+t)}-p_1(x_0)\right\}=0.
\end{aligned}\right\} \tag{60}$$

The proof of this is so far similar in character to the argumentation associated with equations (55) to (58) that it may readily be supplied by the reader. Furthermore, any solution of (2) having the properties (60) is easily seen to be identical with the solution $f(x)$ defined by (59). Consequently, the relations (60) may be looked upon as a sort of set of "initial conditions" at infinity, defining uniquely a solution of the given equation (2).

In general these relations are complicated. It may be observed, however, that the single "initial condition"

* The asymptotic character of the first solution $\bar{f}_1^+(x)$ is maintained with respect to x in any unrestricted right half-plane.

$$\lim_{t=\infty} t^m\left\{\frac{f(x_0+t)}{l_1(x_0+t)}-p(x_0)\right\}=0$$

is sufficient to define uniquely a solution $f(x)=p(x)\overline{f}_1^+(x)$. In this case alone is the solution defined in a simple way by the initial conditions.

§ 4. *Second System of Intermediate Solutions.*

The fundamental set of solutions of the preceding section was obtained by means of a class of transformations depending on the solution $f_1^+(x)$ of § 2. If one starts from the solution $f_n^-(x)$ of § 2 and proceeds by a method in every respect similar to that employed in § 3 he will obviously be led to a new fundamental system of solutions related to those in § 3 as $f_n^-(x)$ is related to $f_1^+(x)$. It is unnecessary to give the argumentation. We merely state the result:

Second System of Intermediate Solutions. *If the characteristic algebraic equation associated with the difference equation* (2) *has its roots different from each other and from zero, then the difference equation has a fundamental system of solutions,*

$$\overline{f}_1^-(x),\ \overline{f}_2^-(x),\ \ldots,\ \overline{f}_n^-(x),$$

possessing the following properties:

1) *A constant V exists such that each function in the set is analytic if $|v|\geqq V$, where v is defined by writing $x=u+v\sqrt{-1}$, u and v being real;*

2) *The asymptotic relations*

$$\overline{f}_i^-(x)\sim\alpha_i^x x^{\mu_i}\left(1+\frac{c_{i1}}{x}+\frac{c_{i2}}{x^2}+\ldots.\right),\qquad i=1,2,\ldots.,n,$$

exist and are valid with respect to x in that part of any left half-plane for which $|v|\geqq V$ and with respect to v in the entire plane.*

The second property implies a set of relations analogous to those in (54). In fact, it is by means of such relations that this part of the theorem is demonstrated. From these follows a set of "initial conditions" analogous to those in (60), the limits in the present case being taken for t approaching $-\infty$. The reader may readily supply them.

§ 5. *The First System of Principal Solutions.*

By means of the particular solution $f_1^+(x)$ of equation (2), obtained in § 2, we proceeded in § 3 to build up a fundamental set of solutions. In doing this

* The asymptotic character of $\overline{f}_n^-(x)$ is maintained with respect to x in any unrestricted left half-plane.

we made a transformation which introduced a new function $g_1^+(x)$ satisfying the relation

$$\Delta g_1^+(x)=G_1^+(x), \tag{43 bis}$$

$G_1^+(x)$ being a particular solution of equation (38), this latter equation being obtained from equation (2) by means of the transformation (36). For $g_1^+(x)$ we took the function $\bar{g}_1^+(x)$, defined in equation (44), this function satisfying equation (43). We stated that another solution of (43) would also lead to interesting solutions of (2). These latter solutions we shall now find.

Throughout this discussion we shall assume that the integer m has been so chosen that $f_1^+(x)$ is independent of m, this being possible, as was shown in § 3. Then we have not merely the limits affecting $f_1^+(x)$, which were stated in the preliminary theorem of § 2, but also the further property expressed in the asymptotic relation

$$f_1^+(x)\sim\alpha_1^x x^{\mu_1}\left(1+\frac{c_{11}}{x}+\frac{c_{12}}{x^2}+\dots\right),$$

valid with respect to x in the entire right half-plane and with respect to v in the entire plane. Similarly, if we make use of this extended property of $f_1^+(x)$, then, instead of the inequality (42), we have the asymptotic relation

$$G_1^+(x)\sim\left(\frac{\alpha_2}{\alpha_1}\right)^x x^{\mu_2-\mu_1}\left(\frac{\alpha_2}{\alpha_1}-1\right)\left(1+\frac{d_{11}}{x}+\frac{d_{12}}{x^2}+\dots\right),$$

valid with respect to x in the entire right half-plane and with respect to v in the entire plane. Furthermore, the function $\bar{g}_1^+(x)$, defined by equation (44), is asymptotically represented by

$$\bar{g}_1^+(x)\sim\left(\frac{\alpha_2}{\alpha_1}\right)^x x^{\mu_2-\mu_1}(1+\dots)\equiv\frac{\alpha_2^x x^{\mu_2}(1+c_{21}x^{-1}+\dots)}{\alpha_1^x x^{\mu_1}(1+c_{11}x^{-1}+\dots)},$$

as one sees readily from its properties as developed in § 3; the range being the same as before, except that $|v|$ must now be restricted to be not less than a certain fixed quantity.

A solution $g_1^+(x)$ of (43) in the form of a contour integral, the path of integration going to infinity along two parallel lines, will be found to serve our purpose. We form the function

$$g_1^+(x)=\int_L\rho(x-z)\,p(x,z)\,G_1^+(z)\,dz, \tag{61}$$

where

$$\rho(x-z)=\{1-e^{2\pi\sqrt{-1}(x-z)}\}^{-1},$$

and where the path of integration L is yet to be chosen and the function $p(x,z)$ is to be determined subject to the condition that it is analytic with respect to x

and also with respect to z throughout the finite planes of these variables and that

$$p(x+1, z) = p(x, z).$$

Let RD be a right D-region throughout which $f_1^+(x)$ and $G_1^+(x)$ are both analytic and verify the relations (31′) and (42), respectively, for every value of m. Since $G_1^+(x)$ is analytic throughout RD, it is clear that all the infinities of the integrand in (61) are at the points $z = x \pm r$, where r is zero or a positive integer. Let AB be the straight line containing all these points. Let the path of integration L or $CKRH$ lie entirely within RD and be formed in the following manner: It consists of three parts: a part RH lying entirely above AB and extending to infinity in a negative direction RH parallel to the axis of reals; a part KC lying below AB and extending to infinity in a negative direction KC parallel to the axis of reals; a finite part KR which crosses AB once between $x-1$ and x and at no other point. Moreover, the path $CKRH$ is such that the part of the finite plane not in RD lies entirely to the left of this path.

In order that the integral along such a path shall exist, it is necessary that the function $p(x, z)$ shall have certain requisite properties. The value $p(x, z) = 1$ would satisfy this requirement, provided that $|\alpha_1| < |\alpha_2|$; and it would indeed lead to solutions of our equation (2) (compare my memoir, *loc. cit.*, p. 119). Simpler solutions may, however, be obtained by a different choice (compare Birkhoff, *loc. cit.*, p. 264).

For the present we assume that $|\alpha_1| < |\alpha_2|$ and choose for $p(x, z)$ the function

$$p(x, z) = e^{2\pi\lambda\sqrt{-1}(x-z)},$$

where λ is the least integer not less than the real part of

$$\frac{1}{2\pi\sqrt{-1}}(\log \alpha_2 - \log \alpha_1).$$

Then for $g_1^+(x)$ we have the value

$$g_1^+(x) = \int_L \rho(x-z)\, e^{2\pi\lambda\sqrt{-1}(x-z)} G_1^+(z)\, dz. \tag{62}$$

It is evident that this function is analytic throughout RD. Furthermore, the value of $g_1^+(x)$, for a particular x, is unaffected by any deformation of the path of integration, provided that during the deformation it has always the properties stated in its definition.

We show that $g_1^+(x)$, as defined in (62), is indeed a solution of equation (43). We have

$$g_1^+(x+1) = \int_{L'} \rho(x-z)\, e^{2\pi\sqrt{-1}\lambda(x-z)} G_1^+(z)\, dz,$$

where L' is a path $CKTSRH$ which differs from $CKRH$ only in crossing AB between x and $x+1$ instead of between $x-1$ and x. Then

$$\int_{L'}-\int_L=\int_{KTSRK}.$$

By Cauchy's theorem on residues the last integral is equal to $2\pi\sqrt{-1}$ times the residue of the integral at the point $z=x$; that is, it is equal to $G_1^+(x)$. Hence,

$$g_1^+(x+1)-g_1^+(x)=G_1^+(x);$$

or, $g_1^+(x)$ satisfies equation (43). It is this particular solution of (43) which we shall denote by $g_1^+(x)$ in the remaining discussion.

We shall now determine a bound to the magnitude of $|g_1^+(x)|$ for certain ranges of variation for x. For this purpose it is convenient to break up the path of integration into two parts L_1 and L_2, as follows: The contour L_1 is the fixed contour $\infty A_1B_1\infty$ lying entirely in the region RD and containing within it that part of the finite plane which is exterior to RD. Moreover, it consists of three parts, each of which is a straight line, a finite part A_1B_1 perpendicular to the axis of reals, and two infinite parts $A_1\infty$ and $B_1\infty$ parallel to the axis of reals. The contour L_2 lies entirely without the region inclosed by L_1. If x lies above $B_1\infty$, L_2 consists of a loop circuit to infinity lying above $B_1\infty$ and including within it the points $x-1, x-2, x-3, \ldots$, but not the points $x, x+1, x+2, \ldots$. Moreover, in the neighborhood of infinity the bounding lines of L_2 are parallel to the axis of reals. If x lies below $A_1\infty$, L_2 consists of a loop circuit to ∞ of the same nature as when x lies above $B_1\infty$. If x lies between $B_1\infty$ and $A_1\infty$, then L_2 consists of a finite loop containing within it the points $x-1, x-2, \ldots, x-l$, but no other point congruent to x, $x-l$ being the leftmost point congruent to x and not on or within the contour L_1. For the integrand in (62) it is clear that the integral along L is equal to the sum of two integrals, one along L_1 and the other along L_2, provided that x is not congruent to a point on the contour L_1. In symbols

$$I_L=I_{L_1}+I_{L_2},$$

where I_L denotes the integral in (62) and I_{L_1} and I_{L_2} similar integrals taken along the paths L_1 and L_2, the integrand being the same in all three cases.

Let us consider first the integral I_{L_1} It denotes a periodic function of x of period 1 since the contour is fixed and the integrand is a periodic function of x of period 1. It may be written in either of the forms:

$$I_{L_1}=e^{2\pi\lambda\sqrt{-1}x}\int_{L_1}\frac{e^{-2\pi\lambda\sqrt{-1}z}G_1^+(z)\,dz}{1-e^{2\pi\sqrt{-1}(x-z)}},$$

$$I_{L_1}=e^{2\pi(\lambda-1)\sqrt{-1}x}\int_{L_1}\frac{e^{-2\pi(\lambda-1)\sqrt{-1}z}G_1^+(z)\,dz}{e^{-2\pi\sqrt{-1}(x-z)}-1}.$$

If $x, =u+v\sqrt{-1}$, approaches infinity so that v becomes positively infinite, the integral in the first one of these equations is bounded. If v becomes negatively infinite, the integral in the second one of these equations is bounded. If x approaches infinity while v remains bounded, then I_{L_1} is itself bounded. Hence,

$$I_{L_1}=\begin{cases} e^{2\pi\lambda\sqrt{-1}x}q(x), \\ q(x), \\ e^{2\pi(\lambda-1)\sqrt{-1}x}q(x), \end{cases}$$

according as x lies above $B_1\infty$, between $B_1\infty$ and $A_1\infty$, or below $A_1\infty$, the function $q(x)$ in each case being periodic in x of period 1 and bounded. This result is strictly valid only when x is not congruent to a point of the contour L_1. For all such x as this, it is obvious that a single modified contour might be used and that a similar form for I_L would be obtained, the periodic functions $q(x)$ being different in this case, but still being bounded.

Let us next consider the integral I_{L_2}. We separate the treatment into two cases.

In the first place, let us suppose that x lies above $B_1\infty$ or below $A_1\infty$. In this case the integral I_{L_2} may be evaluated as a sum of residues multiplied by $2\pi\sqrt{-1}$; thus, we have

$$I_{L_2}=G_1^+(x-1)+G_1^+(x-2)+G_1^+(x-3)+\ldots\ldots$$

Hence, in this case,

$$I_{L_2}=\bar{g}_1^+(x),$$

a function whose asymptotic character is known.

In the second place, let us suppose that x lies between $B_1\infty$ and $A_1\infty$ and to the right of A_1B_1. Then, evaluating as before by means of residues, we have

$$I_{L_2}=\sum_{k=1}^{l} G_1^+(x-k).$$

The asymptotic character of I_{L_2} with respect to x in the right half-plane can now be determined in precisely the same way as that by which the corresponding bounds to the increase of $|\bar{g}_1^+(x)|$ were determined in § 3. The only modification necessary arises from the fact that we now have a finite series instead of the infinite series in (44), so that when this series is separated into two parts, as in the earlier discussion, the second part as well as the first consists of only a finite number of terms. But it was precisely this second part which did not affect the asymptotic form, so that the modification in the argument will not affect the conclusion.

Taking these two cases together, then, and remembering the asymptotic form of $\bar{g}_1^+(x)$, we find that

$$I_{L_2} \sim \left(\frac{\alpha_2}{\alpha_1}\right)^x x^{\mu_2-\mu_1}(1+\ldots.) \equiv \frac{\alpha_2^x x^{\mu_2}(1+c_{21}x^{-1}+\ldots.)}{\alpha_1^x x^{\mu_1}(1+c_{11}x^{-1}+\ldots.)}$$

throughout the right half-plane.

Now let us define a second solution $f_2^+(x)$ of equation (2) by means of the relation

$$f_2^+(x) = g_1^+(x) f_1^+(x) = (I_{L_1}+I_{L_2}) f_1^+(x).$$

From the known form of I_{L_1} and the known asymptotic character of I_{L_2} and $f_1^+(x)$ we see that

$$\left.\begin{aligned} f_2^+(x) &\sim e^{2\pi\lambda\sqrt{-1}x} q(x)\left\{\alpha_1^x x^{\mu_1}\left(1+\frac{c_{11}}{x}+\ldots.\right)\right\} + \alpha_2^x x^{\mu_2}\left(1+\frac{c_{21}}{x}+\ldots.\right), \\ f_2^+(x) &\sim q(x)\left\{\alpha_1^x x^{\mu_1}\left(1+\frac{c_{11}}{x}+\ldots.\right)\right\} + \alpha_2^x x^{\mu_2}\left(1+\frac{c_{21}}{x}+\ldots.\right), \\ f_2^+(x) &\sim e^{2\pi(\lambda-1)\sqrt{-1}x} q(x)\left\{\alpha_1^x x^{\mu_1}\left(1+\frac{c_{11}}{x}+\ldots.\right)\right\} + \alpha_2^x x^{\mu_2}\left(1+\frac{c_{21}}{x}+\ldots.\right), \end{aligned}\right\} \quad (63)$$

according as x lies above $B_1\infty$, between $B_1\infty$ and $A_1\infty$ or below $A_1\infty$, the asymptotic representation being valid with respect to x in any right half-plane and with respect to v in the entire plane.

From this result we may determine, in simpler form, the asymptotic character of $f_2^+(x)$ with respect to x in a right half-plane. Suppose first that x lies above $B_1\infty$. The dominating term in the above representation of $f_2^+(x)$ depends upon the dominating term of the set

$$e^{(2\pi\lambda\sqrt{-1}+\log \alpha_1)x}, \quad e^{(\log \alpha_2)x}.$$

If we divide both of these quantities by the last, the exponents are

$$2\pi\sqrt{-1}\left\{\lambda + \frac{\log \alpha_1 - \log \alpha_2}{2\pi\sqrt{-1}}\right\} x, \quad 0.$$

If the real part of the first exponent is negative the latter of the two quantities predominates. From the definition of λ and the fact that we now have $|\alpha_1| < |\alpha_2|$ it follows readily that the real part of this first exponent is indeed negative if u (where $x = u + v\sqrt{-1}$) is positive and also if u is merely bounded below, provided that the real part of $(\log \alpha_2 - \log \alpha_1)/(2\pi\sqrt{-1})$ is not the integer λ and v is sufficiently large. Therefore,

$$f_2^+(x) \sim \alpha_2^x x^{\mu_2}\left(1+\frac{c_{21}}{x}+\ldots.\right) \quad (64)$$

for x above $B_1\infty$ and to the right of the axis of imaginaries; and, unless the real part of $(\log\alpha_2-\log\alpha_1)/(2\pi\sqrt{-1})$ is an integer, the asymptotic representation is valid for x above $B_1\infty$ and to the right of any line parallel to the axis of imaginaries.

Suppose next that x lies below $A_1\infty$. The dominating term in the asymptotic representation (63) of $f_2^+(x)$ now depends upon the dominating term of the set

$$e^{\{2\pi(\lambda-1)\sqrt{-1}+\log\alpha_1\}x},\quad e^{(\log\alpha_2)x}.$$

The second of these predominates, provided that the real part of

$$2\pi\sqrt{-1}\left\{\lambda-1+\frac{\log\alpha_1-\log\alpha_2}{2\pi\sqrt{-1}}\right\}x$$

is negative; and this real part is in fact negative if u is positive or if u is merely bounded below and $-v$ is sufficiently large. Hence, the asymptotic representation (64) is valid for x below $A_1\infty$ and to the right of any line parallel to the axis of imaginaries.

Finally, suppose that x lies between $A_1\infty$ and $B_1\infty$. Then the dominating term in the asymptotic representation (63) of $f_2^+(x)$ is clearly the last term, since now $|\alpha_1|<|\alpha_2|$, so that the asymptotic representation (64) is valid for x between $A_1\infty$ and $B_1\infty$.

Combining these results, we see that the asymptotic representation (64) is certainly valid with respect to x for x approaching infinity in any way so as to remain to the right of the axis of imaginaries when in the upper half-plane and to the right of any line parallel to the axis of imaginaries when in the lower half-plane. Moreover, if the real part of $(\log\alpha_2-\log\alpha_1)/(2\pi\sqrt{-1})$ is not an integer, then the asymptotic representation is valid with respect to x in any right half-plane.

The above result was obtained on the supposition that $|\alpha_1|<|\alpha_2|$. If $|\alpha_1|=|\alpha_2|$, we form a second solution $f_2^+(x)$ in the same way as $f_1^+(x)$ was formed in § 2 (see the preliminary theorem), this being obviously possible since α_1 and α_2 then play exactly parallel rôles. This solution $f_2^+(x)$ has the asymptotic representation (64) with respect to x in any right half-plane.

Thus, in any event, we have two solutions of equation (2). We proceed further as in § 3. Equation (38) is now known to have a second solution and its asymptotic character is determined in the same way as that of $f_2^+(x)$. By means of this second solution of equation (38) we are led to a third solution of equation (2). Then by a further reaction between equations (2) and (38) we find another solution of (2), and so on until n solutions of (2) are found.

These are analytic in a suitable right half-plane and are represented asymptotically by the formal power series solutions of (2). By means of their asymptotic form it may be shown (as in the case of the solutions obtained in § 3) that they are linearly independent with respect to periodic multipliers of period 1, and hence form a fundamental system of solutions of equation (2). Since they are analytic in a suitable right half-plane, the possible position of their singularities may be determined as in the case of $f_1^+(x)$ in § 2. Combining the various results which we thus have, we obtain the following theorem:

FIRST SYSTEM OF PRINCIPAL SOLUTIONS. *If the characteristic algebraic equation associated with the difference equation* (2) *has its roots different from each other and from zero, then this difference equation has the fundamental system of solutions*

$$f_1^+(x),\, f_2^+(x),\, \ldots,\, f_n^+(x),$$

determined above and possessing the following properties:

1) *Each function in the system is analytic throughout the finite plane, except at the singularities of the functions* (34) *and points congruent to them on the left. Moreover, the singularities, in the finite plane, of each function in the solution are poles, provided that* $a_1(x), a_2(x), \ldots, a_n(x)$ *are rational functions and the complete set of numbers, each of which is the order of one of these poles, is a bounded set.*

2) *The functions* $f_i^+(x)$, $i=1, 2, \ldots, n$, *have with respect to* x *the asymptotic representation*

$$f_i^+(x) \sim \alpha_i^x x^{\mu_i}\left(1+\frac{c_{i1}}{x}+\frac{c_{i2}}{x^2}+\ldots\right), \qquad i=1, 2, \ldots, n,$$

this representation being valid for x *approaching infinity in any way so as to remain to the right of the axis of imaginaries when in the upper half-plane and to the right of any line parallel to the axis of imaginaries when in the lower half-plane.* Moreover, this representation is valid in any right half-plane provided that no one of the quantities*

$$\frac{\log \alpha_i - \log \alpha_j}{2\pi\sqrt{-1}}, \qquad j=1, 2, \ldots, n;\ i=j+1, \ldots, n,$$

has its real part equal to an integer.

* The upper and lower half-planes enter here unsymmetrically. Their rôles may be interchanged by a different choice of the quantities λ entering into the discussion. See the paragraph containing equation (62).

From results previously obtained we know, furthermore, that $f_1^+(x)$ always maintains its asymptotic character with respect to x in any right half-plane and with respect to v in the entire plane.

The solutions mentioned in the foregoing theorem are called principal solutions on account of the important properties which they possess.

If $f_k(x)$ is any solution of (2) which is analytic in some right half-plane and is there represented asymptotically with respect to x by

$$\alpha_k^x x^{\mu_k}\left(1+\frac{c_{k1}}{x}+\frac{c_{k2}}{x^2}+\cdots\right),$$

then it may be shown (as in a similar case in § 3, beginning with equation (55)) that*

$$f_k(x)=p_1(x)f_1^+(x)+\cdots+p_{k-1}(x)f_{k-1}^+(x)+f_k^+(x),$$

where $p_1(x),\ldots, p_{k-1}(x)$ are suitably determined periodic functions of x of period 1. By means of this result one again has a theory of "initial conditions" analogous to that associated with equation (60).

§ 6. *The Second System of Principal Solutions.*

In the preceding section a fundamental system of solutions was obtained by starting with the solution $f_1^+(x)$, obtained in § 2, and carrying out a certain process of transformation of equation (2) and interaction between it and the transformed equation. Similarly, one may start from the solution $f_n^-(x)$, obtained in § 2, and by a similar process arrive at another fundamental system of solutions of equation (2). We state merely the result of this work in the form of the following theorem:

Second System of Principal Solutions. *If the characteristic algebraic equation associated with the difference equation* (2) *has its roots different from each other and from zero, then this difference equation has a fundamental system of solutions*

$$f_1^-(x),\ f_2^-(x),\ \ldots,\ f_n^-(x),$$

possessing the following properties:

1) *Each function in the system is analytic throughout the finite plane except at points congruent on the right* (*at a distance of n units or more*) *to the singularities of the functions $a_1(x)$, $a_2(x),\ldots, a_n(x)$. Moreover, the singularities of each solution, in the finite plane, are poles provided that $a_1(x)$,*

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* If $f_k(x)$ is replaced by the k-th solution of the first fundamental set of associated solutions, then it is easy to see that we continue to have the same relation as that given in the text, this relation certainly being valid if $|v|$ is sufficiently large.

$a_2(x), \ldots, a_n(x)$ are rational functions and the complete set of numbers, each of which is the order of one of these poles, is a bounded set.

2) *The functions $f_i^-(x)$, $i=1, 2, \ldots, n$, have with respect to x the asymptotic representation*

$$f_i^-(x) \sim \alpha_i^x x^{\mu_i}\left(1+\frac{c_{i1}}{x}+\frac{c_{i2}}{x^2}+\ldots\right), \qquad i=1, 2, \ldots, n,$$

this representation being valid for x approaching infinity in any way so as to remain to the left of the axis of imaginaries when in the lower half-plane and to the left of any line parallel to the axis of imaginaries when in the upper half-plane. Moreover, this representation is valid in any left half-plane, provided that no one of the quantities*

$$\frac{\log \alpha_i - \log \alpha_j}{2\pi\sqrt{-1}}, \; j=1, 2, \ldots, n; \; i=j+1, \ldots, n,$$

has its real part equal to an integer.

Further remarks, similar to those at the close of the preceding section, may be made with reference to the solutions $f_1^-(x), \ldots, f_n^-(x)$.

§ 7. *Other Results.*

The methods by which the foregoing results are demonstrated are essentially distinct from any at present existent in the literature of difference equations. There are three other results which are necessary to complete the first fundamentals of a general theory of equation (1). These are due to Birkhoff (*loc. cit.*), who has demonstrated them for a system of n equations of the first order rather than for a single equation of order n. Birkhoff's proof, with merely formal modifications, will apply to equation (1) and may be based on the theorems demonstrated above. It is therefore unnecessary to repeat the proof. It seems desirable, however, to have a careful statement of these in a form to be directly applicable to equation (1) and for the sake of completeness of the fundamental results assembled in this paper. Consequently, a statement of these theorems, without proof, is given below:

1. If the characteristic algebraic equation associated with the difference equation (1) has its roots different from each other and from zero, and if the coefficients $\bar{a}_1(x), \bar{a}_2(x), \ldots, \bar{a}_n(x)$ are polynomials in x and if further we write

* Compare the foot-note to the previous theorem.

$$F_i^-(x) = p_{i1}(x)\, F_1^+(x) + p_{i2}(x)\, F_2^+(x) + \dots + p_{in}(x)\, F_n^+(x), \quad i=1, 2, \dots, n,$$

where

$$F_i^+(x) = x^{\mu x} f_i^+(x), \quad F_i^-(x) = x^{\mu x} f_i^-(x), \quad i=1, 2, \dots, n,$$

then the fundamental periodic functions $p_{ij}(x)$ are of the form

$$p_{ii}(x) = 1 + c_{ii}^{(1)} e^{2\pi \sqrt{-1}\, x} + c_{ii}^{(2)} e^{4\pi \sqrt{-1}\, x} + \dots + e^{2\pi \mu_i \sqrt{-1}}\, e^{2\pi\mu \sqrt{-1}\, x},$$
$$p_{ij}(x) = e^{2\pi\lambda_{ij} \sqrt{-1}\, x}[c_{ij}^{(0)} + c_{ij}^{(1)} e^{2\pi \sqrt{-1}\, x} + \dots + c_{ij}^{(\mu-1)} e^{2(\mu-1)\pi \sqrt{-1}\, x}], \quad i \neq j,$$

where λ_{ij} denotes the least integer not less than the real part of

$$\frac{\log \alpha_i - \log \alpha_j}{2\pi\sqrt{-1}}.$$

If the functions $\bar{a}_1(x)$, $\bar{a}_2(x)$, $\dots$, $\bar{a}_n(x)$ are restricted merely to be rational, then the periodic functions $p_{ij}(x)$ are rational functions of $e^{2\pi\sqrt{-1}\, x}$.

2. Let the characteristic algebraic equation associated with the difference equation (1) have its roots different from each other and from zero, and let the coefficients $\bar{a}_1(x)$, $\bar{a}_2(x)$, $\dots$, $\bar{a}_n(x)$ be polynomials in x. Write

$$2\pi\sqrt{-1}\left(\frac{\log \alpha_j - \log \alpha_i}{2\pi\sqrt{-1}} + \lambda_{ij}\right) = \varepsilon_{ij},$$

where λ_{ij} is the least integer not less than the real part of the negative of the first term in parenthesis. Mark the two sets of points

$$\varepsilon_{in},\ \varepsilon_{i,n-1},\ \dots,\ \varepsilon_{i,i-1},\ 0 \text{ and } \varepsilon_{in} - 2\pi\sqrt{-1},\ \dots,\ \varepsilon_{i,i-1} - 2\pi\sqrt{-1},\ 0,$$

and call them, respectively,

$$P_n,\ P_{n-1},\ \dots,\ P_i \text{ and } Q_n,\ Q_{n-1},\ \dots,\ Q_i.$$

Construct the convex broken line $P_n\, P_\sigma \dots P_i$ above which all the remaining points P lie, and likewise construct the convex broken line $Q_n\, Q_\theta \dots Q_i$ below which all the remaining points Q lie. Let the acute angles which the successive sides of $P_n P_\sigma \dots P_i$ and $Q_n Q_\theta \dots Q_i$ make with the axis of reals be $\phi_n, \phi_\sigma, \dots$ and $\psi_n, \psi_\theta, \dots$, respectively. The critical rays between $\arg x = 0$ and $\arg x = 2\pi$ are then

$$\frac{\pi}{2} - \phi_n,\ \frac{\pi}{2} - \phi_\sigma,\ \dots,\ -\frac{\pi}{2} + \psi_\theta,\ -\frac{\pi}{2} + \psi_n,$$

in angular order. The asymptotic form of $F_i^-(x)$ changes from

$$e^{2\pi\lambda_{it}\sqrt{-1}\, x}\, c_{it}^{(0)}\, x^{\mu x}\, \alpha_t^x\, x^{\mu_t}\left(1 + \frac{c_{t1}}{x} + \frac{c_{t2}}{x^2} + \dots\right)$$

to

$$e^{2\pi\lambda_{is}\sqrt{-1}\, x} c_{is}^{(0)} x^{\mu x} \alpha_s^x x^{\mu_s}\left(1 + \frac{c_{s1}}{x} + \frac{c_{s2}}{x^2} + \dots\right),$$

along the critical ray $\arg x = \frac{1}{2}\pi - \phi_t$, and likewise from*

$$e^{2\pi(\lambda_{tt}+\delta_{tt}-1)\sqrt{-1}x} c_{it}^{(\mu+\delta_{it}-1)} e^{-2\pi\mu_t\sqrt{-1}} x^{\mu x} \alpha_t^x x^{\mu_t}\left(1+\frac{c_{t1}}{x}+\ldots\right),$$

to

$$e^{2\pi(\lambda_{is}+\delta_{is}-1)\sqrt{-1}x} c_{is}^{(\mu+\delta_{is}-1)} e^{-2\pi\mu_s\sqrt{-1}} x^{\mu x} \alpha_s^x x^{\mu_s}\left(1+\frac{c_{s1}}{x}+\ldots\right),$$

along the critical ray $\arg x = -\frac{1}{2}\pi + \psi_t$. Between the last critical ray of the first set and the first of the last set the asymptotic form is given by

$$x^{\mu x} \alpha_i^x x^{\mu_i}\left(1+\frac{c_{i1}}{x}+\ldots\right).$$

A similar theorem exists for the functions $F_i^+(x)$ in case equation (1) is such that it may be written in the form

$$a_0'(x)F(x+n)+\ldots+a_{n-1}'(x)F(x+1)+F(x)=0,$$

where $a_0'(x)$, $a_1'(x)$, ..., $a_{n-1}'(x)$ are polynomials

3. Let $F_1^+(x)$, ..., $F_n^+(x)$ and $F_1^-(x)$, ..., $F_n^-(x)$ be two sets of single-valued functions which are analytic throughout the finite plane, except for poles, and which have the further property that

$$\lim_{x=\infty} F_i^+(x)x^{-\mu x}\alpha_i^{-x}x^{-\mu_i}=1, \quad \lim_{x=\infty} F_i^-(x)x^{-\mu x}\alpha_i^{-x}x^{-\mu_i}=1,$$

μ being an integer, the constants $\alpha_1, \alpha_2, \ldots, \alpha_n$ being different from each other and from zero and the range of x for the limits being as follows: In the first [second] case x may approach infinity in any way so as to remain to the right [left] of any line parallel to the axis of imaginaries when in the upper [lower] half-plane and to the right [left] of any line parallel to the axis of imaginaries when in the lower [upper] half-plane. Furthermore, let these two sets of functions be connected by the relations

$$F_i^-(x)=p_{i1}(x)F_1^+(x)+\ldots+p_{in}(x)F_n^+(x), \qquad i=1, 2, \ldots, n,$$

where the functions $p_{ij}(x)$ are periodic in x of period 1. Then the sets $F_1^+(x)$, ..., $F_n^+(x)$ and $F_1^-(x)$, ..., $F_n^-(x)$ are fundamental systems of solutions of a difference equation (1) in which the coefficients $\bar{a}_1(x)$, ..., $\bar{a}_n(x)$ are rational functions of x.

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* Here δ_{ij} is equal to unity or zero, according as i is or is not equal to j.